

Hawking's singularity theorem for $C^{1,1}$ -metrics

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Abstract

We provide a detailed proof of Hawking's singularity theorem in the regularity class $C^{1,1}$, i.e., for spacetime metrics possessing locally Lipschitz continuous first derivatives. The proof uses recent results in $C^{1,1}$ -causality theory and is based on regularisation techniques adapted to the causal structure.

Keywords: Singularity theorems, low regularity, regularisation, causality theory

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1 Introduction

In the early years of General Relativity it was known that there existed solutions of the Einstein field equations which had singular behaviour of various kinds. However, the prevailing view was that these singularities were the result of the high degree of symmetry or were unphysical in some way. This position changed considerably with the work of Penrose who showed in his 1965 paper [27] that deviations from spherical symmetry could not prevent gravitational collapse. This paper not only introduced the concept of closed trapped surface, but used the notion of geodesic incompleteness to characterise a singular spacetime.

Shortly afterwards Hawking realised that by considering a closed trapped surface to the past one could show that an approximately homogeneous and isotropic cosmological solution must have an initial singularity. There quickly followed a series of papers by Hawking, Penrose, Ellis, Geroch and others which led to the development of modern singularity theorems, one of the greatest achievements within general relativity. (See the recent review paper [29] for details.) The resulting theorems all had the same general framework described by Senovilla in [28] as a “pattern singularity theorem”.

Pattern Singularity Theorem. *If a spacetime with a C^2 -metric satisfies*

- (i) *a condition on the curvature*
- (ii) *a causality condition*
- (iii) *an appropriate initial and/or boundary condition*

then it contains endless but incomplete causal geodesics.

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Despite their power and glory the singularity theorems have a weak point, which is their conclusion. In fact, they simply show causal geodesic incompleteness of the spacetime but say little about the nature of the singularity. In particular, they do not say that the curvature blows up (see, however [3, 4] as well as [29, Sec. 5.1.5] and the references therein) and it could be that the singularity is simply a result of the differentiability dropping below C^2 . In the case that the regularity of the metric simply dropped to $C^{1,1}$ (also denoted by C^{2-} , the first derivatives of the metric being locally Lipschitz continuous) the theorems would predict the curvature to become discontinuous rather than unbounded. Recall that indeed the connection of a $C^{1,1}$ -metric is locally Lipschitz and hence by Rademacher's theorem differentiable almost everywhere with locally bounded curvature. From the viewpoint of physics such a situation would hardly be regarded as 'singular' as it corresponds, via the field equations, to a finite jump in the matter variables. There are many physically realistic systems of that type, such as the Oppenheimer-Snyder model of a collapsing star [26], to give a classical example, and general matched spacetimes, see e.g. [18, 19].

Also from the point of view of the singularity theorems themselves the natural differentiability class is $C^{1,1}$. Indeed this is the minimal condition which ensures existence and uniqueness of solutions of the geodesic equation, which is essential to the statement of the theorems. Moreover, as already pointed out in [13, Sec. 8.4], in the context of a $C^{1,1}$ -singularity theorem a further dropping of the regularity would result in spacetimes where generically the curvature diverges and in addition there are problems with the uniqueness of causal geodesics and hence the worldlines of physical observers. Such a situation could be interpreted as physically 'singular' with much better reason than the corresponding C^2 -situation discussed above.

All this provides a strong motivation for trying to prove the singularity theorems in the regularity class $C^{1,1}$. In [13, Sec. 8.4] Hawking and Ellis discuss the nature of the singularities predicted by the singularity theorems and go on to outline a proof of Hawking's singularity theorem based on an approximation of the $C^{1,1}$ -metric by a 1-parameter family of smooth metrics. However the C^2 -differentiability assumption plays a key role in many places in the singularity theorems and it is not obvious that these can all be dealt with without having further information about the nature of the approximation. Indeed much of standard causality theory assumes that the metric is smooth or at least C^2 , see e.g. [13, 23, 4, 28, 9, 6, 30] for a review of various approaches to causal structures and discussions of the regularity assumptions. Senovilla in [28, Sec. 6.1] lists those places where the C^2 -assumption explicitly enters the proofs of the singularity theorems, indicating the number of technical difficulties a proof in the $C^{1,1}$ -case would have to overcome. Indeed, to our knowledge, the only results that are available in $C^{1,1}$ -singularity theory are very limited [2, 3, 4] or restricted to special situations [20] and we think it is fair to say (cf. [28]) that the issue of regularity in the singularity theorems is often ignored despite its mathematical and physical relevance.

Motivated by the physical arguments given above and recent advances in the regularity required for the initial value problem (see e.g. [15]) there has been an increased interest in causality theory of spacetimes of low regularity. Chrusciel and Grant in [7] adopted a regularisation approach which is adapted to the causal structure: a given metric of low regularity is approximated by two nets of smooth metrics \tilde{g}_ϵ and \hat{g}_ϵ whose light cones sandwich those of g . They established some fundamental elements of causality theory in low regularity such as the existence of smooth time functions on domains of dependence even for continuous metrics, see also [8]. However, they also revealed a dramatic failure of fundamental results of smooth causality if the regularity was below $C^{1,1}$. In particular, they demonstrated the

existence of ‘bubbling metrics’ (of regularity $C^{0,\alpha}$, for any $\alpha \in (0, 1)$), whose light-cones have nonempty interior, thereby nicely complementing classical examples by Hartman and Wintner [11, 12] which demonstrate the failure of convexity properties in the Riemannian case.

One of the key technical tools employed in causality theory is the exponential map and the existence of totally normal neighbourhoods which allow one to relate the causal structure of Minkowski space to that of the manifold in any given point (see Theorem A.1). Classical results for $C^{1,1}$ -metrics only show that the exponential map is a local homeomorphism [31], which is insufficient to establish the required results. Recently, however, it has been shown that \exp is a bi-Lipschitz homeomorphism. Using a careful analysis of the corresponding ODE problem based on Picard-Lindelöf approximations, as well as an inverse function theorem for Lipschitz maps, Minguzzi [22] established the fact that \exp is a bi-Lipschitz homeomorphism [22, Th. 1.11] and used this to derive many standard results in causality theory. Also the present authors in [16, Th. 2.1] and [17] established similar results by extending the refined regularisation methods of [7] and combining them with methods from comparison geometry [5].

Given that finally the key elements of causality theory are in place, now is the time to approach the singularity theorems for $C^{1,1}$ -metrics. Indeed, in this work we will show that the tools now available allow one to prove singularity theorems with $C^{1,1}$ -regularity and we illustrate this by providing a rigorous proof of Hawking’s theorem in the $C^{1,1}$ -regularity class. To be precise we establish the following result:

Theorem 1.1. *Let (M, g) be a $C^{1,1}$ -spacetime. Assume*

- (i) *For any smooth timelike local vector field X , $\text{Ric}(X, X) \geq 0$.*
- (ii) *There exists a compact spacelike hypersurface S in M .*
- (iii) *The future convergence \mathbf{k} of S is everywhere strictly positive.*

Then M is future timelike geodesically incomplete.

Remark 1.2.

- (i) For the definition of a $C^{1,1}$ -spacetime, see Section 2. Since g is $C^{1,1}$, its Ricci-tensor is of regularity L^∞ . In particular, it is in general only defined almost everywhere. For this reason, we have cast the curvature condition (i) in the above form. For any smooth vector field X defined on an open set $U \subseteq M$, $\text{Ric}(X, X) \in L^\infty(U)$, so $\text{Ric}(X, X) \geq 0$ means that $\text{Ric}_p(X(p), X(p)) \geq 0$ for almost all $p \in U$. Since any timelike $X \in T_p M$ can be extended to a smooth timelike vector field in a neighborhood of p , (i) is equivalent to the usual pointwise condition ($\text{Ric}(X, X) \geq 0$ for any timelike $X \in TM$) if the metric is C^2 .
- (ii) Concerning (iii) in the Theorem, our conventions (in accordance with [25]) are that $\mathbf{k} = \text{tr } S_U / (n - 1)$ and $S_U(V) = -\nabla_V U$ is the shape operator of S , where U is the future pointing unit normal, ∇ denotes the connection on M and V is any vector field on the embedding $S \hookrightarrow M$.
- (iii) In the physics literature, the negative of what we call the future convergence is often denoted as the expansion of S .

- (iv) Finally, we note that an analogous result for past timelike incompleteness holds if the convergence in (iii) of the Theorem is supposed to be everywhere strictly negative.

In proving this theorem we will follow the basic strategy outlined in [13, Sec. 8.4]. However, in our proof we will make extensive use of the recent results of $C^{1,1}$ -causality theory. An important feature of this paper is that we carefully collect all the results from $C^{1,1}$ -causality theory that are required for the proof of the above theorem and show how they can be obtained from [7, 22, 17]. In addition, in section 4 we make crucial use of causal regularisation techniques to show the existence of maximising curves. We therefore need to establish the existence of an approximating family of smooth metrics which satisfy (a weakened form of) the Ricci convergence condition while at the same time controlling the light cones.

The plan of the paper is as follows. In section 2 we fix the definitions and notation we will use in the rest of the paper. In section 3 we introduce the causal regularisation techniques and establish the required estimates for the curvature. In section 4 we make use of the causal regularisation together with some key results from $C^{1,1}$ -causality theory to establish the existence of maximal curves. Finally in section 5 we prove the main result following the basic layout of the proof of [25, Th. 14.55B]. In the Appendix we collect together all the results from causality theory that are required and show how they are proved in the $C^{1,1}$ -case.

2 Preliminaries

In this section we fix key notions to be used throughout this paper. We assume all manifolds to be of class C^∞ (as well as second countable), and only lower the regularity of the metric. This is no loss of generality since any C^k -manifold M with $k \geq 1$ possesses a unique C^∞ -structure that is C^k -compatible with the given C^k -structure on M (see [14, Th. 2.9]). Most of the time (and unless explicitly stated otherwise) we will deal with a $C^{1,1}$ -spacetime (M, g) , by which we mean a smooth manifold M of dimension n endowed with a time-oriented Lorentzian metric g of signature $(- + \cdots +)$ possessing locally Lipschitz continuous first derivatives and with the time orientation given by a continuous timelike vector field. If K is a compact set in M we write $K \Subset M$. Following [25], we define the curvature tensor to be given by $R(X, Y)Z = \nabla_{[X, Y]}Z - [\nabla_X, \nabla_Y]Z$. This convention differs by a sign from that of [13]. We then define the Ricci tensor by $R_{ab} = R^c_{abc}$ (which again differs by a sign from that in [13] where $R_{ab} = R^c_{acb}$, so overall the two definitions of Ricci curvature agree).

There are minor variations in the basic definitions used in causality theory by various authors and this section serves to specify the ones we will be using and relate them to those used elsewhere. Our notation for causal structures will basically follow [25] although following [6, 17] we will base all causality notions on locally Lipschitz curves. We note that in most of the standard literature on causality theory, in particular in [13, 25], the corresponding curves are required to be (piecewise) C^1 . However, as is shown in [22, Th. 1.27], [17, Cor. 3.1], this does not affect the definition of (causal or chronological) pasts and futures. Any locally Lipschitz curve c is differentiable almost everywhere (by Rademacher's theorem) and its derivative is locally bounded. We call c timelike, causal, spacelike or null, if $c'(t)$ has the corresponding property almost everywhere. Based on these notions we define the relative chronological future $I^+(p, A)$ and causal future $J^+(p, A)$ of p in $A \subseteq M$ literally as in the smooth case (see [17, Def. 3.1] [6, Sec. 2.4]). For $B \subseteq A$ we set $I^+(B, A) := \bigcup_{p \in B} I^+(p, A)$ and analogously for $J^+(B, A)$. Moreover, we set $I^+(p) := I^+(p, M)$. The same conventions

apply to the respective past sets where the $+$ is replaced by a $-$. For $p, q \in M$ we write $p < q$, respectively $p \ll q$, if there is a future directed causal, respectively timelike, curve from p to q . By $p \leq q$ we mean $p = q$ or $p < q$. We denote the time separation (distance) between two points $p, q \in M$ and between $A, B \subseteq M$ with respect to some Lorentzian metric g' by $d_{g'}(p, q)$ and $d_{g'}(A, B)$, respectively (cf. [25, Def. 14.15]). We call a $C^{1,1}$ -spacetime (M, g) globally hyperbolic if it is strongly causal and $J(p, q) := J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$. Finally, for an achronal set S , the future Cauchy development of S is the set $D^+(S)$ of all points $p \in M$ with the property that every past inextendible causal curve through p meets S . Then $H^+(S) := \overline{D^+(S)} \setminus I^-(D^+(S))$ is its future Cauchy horizon. Note that both, Cauchy development and Cauchy horizon, are defined with locally Lipschitz causal curves (contrary to [13, 25]). That this does not affect our considerations is shown in Lemma A.12. A Cauchy hypersurface is a subset S of M which every inextendible timelike curve intersects exactly once, see [25, Def. 14.28]. In the smooth case, for spacelike hypersurfaces this definition of a Cauchy hypersurface is equivalent to the one in [13], and this remains true in the $C^{1,1}$ -case, cf. Proposition A.31.

Now let S be a spacelike hypersurface in M with a Lorentzian metric g . By $N(S)$ we denote the set of vectors perpendicular to S with respect to the metric g and by $(N(S), \pi)$ the normal bundle of S in M , where $\pi : N(S) \rightarrow S$ is the map carrying each vector $v \in T_p(S)^\perp$ to $p \in S$. We will distinguish normal bundles stemming from metrics g_ε by writing $(N_{g_\varepsilon}(S), \pi_{g_\varepsilon})$ and for brevity we will drop this subscript for the $C^{1,1}$ -metric g itself. The exponential map with respect to the metric g generalises in the following way: the normal exponential map

$$\exp^\perp : N(S) \rightarrow M$$

assigns to a vector $v \in N(S)$ the point $c_v(1)$ in M , where c_v is the geodesic with initial data v . Thus \exp^\perp carries radial lines in $T_p S$ to geodesics of M that are normal to S at p . Again, in order to distinguish the normal exponential maps w.r.t. metrics g_ε , we write $\exp_{g_\varepsilon}^\perp$. As was shown in [22, Th. 1.39], $N(S)$ is a Lipschitz bundle and \exp^\perp is a bi-Lipschitz homeomorphism from a neighbourhood of the zero section in $N(S)$ onto a neighbourhood of S (cf. Th. A.32 below).

3 Regularisation techniques

While the relevance of regularisation techniques to the problem at hand was already clearly pointed out in [13, Sec. 8.4] we shall see at several places below that a straightforward regularisation via convolution in charts (as in [13, Sec. 8.4]) is insufficient to actually reach the desired conclusions. Rather, techniques adapted to the causal structure as introduced in [7] will be needed. This remark, in particular, applies to the results on the existence of maximising curves (Lemma 4.2 and Proposition 4.3) below as well as to the proof of the main result in Section 5.

Recall from [23, Sec. 3.8.2], [7, Sec. 1.2] that for two Lorentzian metrics g, h , we say that h has *strictly wider light cones* than g , denoted by

$$g \prec h, \text{ if for any tangent vector } X \neq 0, g(X, X) \leq 0 \text{ implies that } h(X, X) < 0. \quad (1)$$

The key result now is [7, Prop. 1.2], which we give here in the slightly refined version of [17, Prop. 2.5]:

Proposition 3.1. *Let (M, g) be a spacetime with a continuous Lorentzian metric, and h some smooth background Riemannian metric on M . Then for any $\varepsilon > 0$, there exist smooth Lorentzian metrics \check{g}_ε and \hat{g}_ε on M such that $\check{g}_\varepsilon \prec g \prec \hat{g}_\varepsilon$ and $d_h(\check{g}_\varepsilon, g) + d_h(\hat{g}_\varepsilon, g) < \varepsilon$, where*

$$d_h(g_1, g_2) := \sup_{p \in M, 0 \neq X, Y \in T_p M} \frac{|g_1(X, Y) - g_2(X, Y)|}{\|X\|_h \|Y\|_h}. \quad (2)$$

Moreover, \hat{g}_ε and \check{g}_ε depend smoothly on ε , and if $g \in C^{1,1}$ then letting g_ε be either \check{g}_ε or \hat{g}_ε , we additionally have

- (i) g_ε converges to g in the C^1 -topology as $\varepsilon \rightarrow 0$, and
- (ii) the second derivatives of g_ε are bounded, uniformly in ε , on compact sets.

One essential assumption in the singularity Theorem 1.1 is the curvature condition (i) for the $C^{1,1}$ -metric g . We now derive from it a (weaker) curvature condition for any approximating sequence \check{g}_ε as in Proposition 3.1, which is vital in our proof of the main theorem. This should be compared to condition (4) on p. 285 of [13].

Lemma 3.2. *Let M be a smooth manifold with a $C^{1,1}$ -Lorentzian metric g and smooth background Riemannian metric h . Let K be a compact subset of M and suppose that $\text{Ric}(X, X) \geq 0$ for every g -timelike smooth local vector field X . Then*

$$\forall C > 0 \ \forall \delta > 0 \ \forall \kappa < 0 \ \exists \varepsilon_0 > 0 \ \forall \varepsilon < \varepsilon_0 \ \forall X \in TM|_K \quad (3)$$

with $g(X, X) \leq \kappa$ and $\|X\|_h \leq C$ we have $\text{Ric}_\varepsilon(X, X) > -\delta$.

Here Ric_ε is the Ricci-tensor corresponding to a metric \check{g}_ε as in Proposition 3.1.

Proof. Let us first briefly recall the notations from the proof of [17, Prop. 2.5]: Let (U_i, ψ_i) ($i \in \mathbb{N}$) be a countable and locally finite collection of relatively compact charts of M and denote by $(\zeta_i)_i$ a subordinate partition of unity with $\text{supp}(\zeta_i) \Subset U_i$ (i.e., $\text{supp}(\zeta_i)$ is a compact subset of U_i) for all i . Moreover, choose a family of cut-off functions $\chi_i \in \mathcal{D}(U_i)$ with $\chi_i \equiv 1$ on a neighbourhood of $\text{supp}(\zeta_i)$. Finally, let $\rho \in \mathcal{D}(\mathbb{R}^n)$ be a non-negative test function with unit integral and define the standard mollifier $\rho_\varepsilon(x) := \varepsilon^{-n} \rho(\frac{x}{\varepsilon})$ ($\varepsilon > 0$). By f_* (resp. f^*) we denote push-forward (resp. pullback) under a smooth map f . It then follows from (2.2) in the proof of [17, Prop. 2.5] that

$$\check{g}_\varepsilon - \sum_i \chi_i \psi_i^* \left((\psi_i)_* (\zeta_i g) \right) * \rho_{\eta(\lambda_i(\varepsilon), i)} \rightarrow 0 \text{ in } C^2(M). \quad (4)$$

Since $\eta(\lambda_i(\varepsilon), i) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and $\{X \in TM|_K \mid \|X\|_h \leq C\}$ is compact, we conclude that in order to establish the result it will suffice to assume that $M = \mathbb{R}^n$, $\|\cdot\|_h = \|\cdot\|$ is the Euclidean norm, to replace \check{g}_ε by $g_\varepsilon := g * \rho_\varepsilon$ (component-wise convolution), and prove (3) for Ric_ε calculated from g_ε .

We first claim that

$$R_{\varepsilon jk} - R_{jk} * \rho_\varepsilon \rightarrow 0 \text{ uniformly on compact sets.} \quad (5)$$

We have $R_{jk} = \partial_{x^i} \Gamma_{kj}^i - \partial_{x^k} \Gamma_{ij}^i + \Gamma_{ij}^i \Gamma_{kj}^m - \Gamma_{km}^i \Gamma_{ij}^m$. In this expression, all terms involving at most first derivatives of g are uniform limits of the corresponding terms in $R_{\varepsilon jk}$, while the

remaining terms are of the form $g^{im}a_{ijkm}$, where a_{ijkm} consists of second derivatives of g . These observations imply that (5) will follow from the following mild variant of the Friedrichs lemma:

Claim: Let $f \in C^0(\mathbb{R}^n)$, $a \in L_{\text{loc}}^\infty(\mathbb{R}^n)$. Then $(f \cdot a) * \rho_\varepsilon - (f * \rho_\varepsilon) \cdot (a * \rho_\varepsilon) \rightarrow 0$ locally uniformly.

In fact,

$$\begin{aligned} \left((f \cdot a) * \rho_\varepsilon - (f * \rho_\varepsilon) \cdot (a * \rho_\varepsilon) \right)(x) &= \int \left(f(y) - (f * \rho_\varepsilon)(x) \right) a(y) \rho_\varepsilon(x - y) dy \\ &= \int \left(f(y) - f(x) \right) a(y) \rho_\varepsilon(x - y) dy + \int \left(f(x) - (f * \rho_\varepsilon)(x) \right) a(y) \rho_\varepsilon(x - y) dy, \end{aligned} \quad (6)$$

so for any $L \subseteq \mathbb{R}^n$ we obtain

$$\begin{aligned} \sup_{x \in L} |(f \cdot a) * \rho_\varepsilon - (f * \rho_\varepsilon) \cdot (a * \rho_\varepsilon)|(x) &\leq \left(\max_{\substack{x \in L \\ |x-y| \leq \varepsilon}} |f(y) - f(x)| \right) \cdot \sup_{d(y,L) \leq \varepsilon} |a(y)| \\ &\quad + \left(\sup_{x \in L} |f(x) - f_\varepsilon(x)| \right) \cdot \sup_{d(y,L) \leq \varepsilon} |a(y)| \rightarrow 0 \end{aligned} \quad (7)$$

as $\varepsilon \rightarrow 0$, so (5) follows.

Since g is uniformly continuous on K there exists some $r > 0$ such that for any $p, x \in K$ with $\|p - x\| < r$ and any $X \in \mathbb{R}^n$ with $\|X\| \leq C$ we have $|g_p(X, X) - g_x(X, X)| < \kappa$. Now let $p \in K$ and let $X \in \mathbb{R}^n$ be any vector such that $g_p(X, X) \leq \kappa$ and $\|X\| \leq C$. Then on the open ball $B_r(p)$ the constant vector field $x \mapsto X$ (i.e., the map that assigns to each $x \in B_r(p)$ this same vector $X \in \mathbb{R}^n$), which we again denote by X , is g -timelike.

Let

$$\tilde{R}_{jk}(x) := \begin{cases} R_{jk}(x) & \text{for } x \in B_r(p) \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

By our assumption and the fact that $\rho \geq 0$ we then have $(\tilde{R}_{jk} X^j X^k) * \rho_\varepsilon \geq 0$ on \mathbb{R}^n . Moreover, for $\varepsilon < r$ it follows that $(R_{jk} * \rho_\varepsilon)(p) = (\tilde{R}_{jk} * \rho_\varepsilon)(p)$. Thus for such ε we have

$$\begin{aligned} |R_{\varepsilon jk}(p) X^j X^k - ((\tilde{R}_{jk} X^j X^k) * \rho_\varepsilon)(p)| &= |(R_{\varepsilon jk}(p) - (R_{jk} * \rho_\varepsilon)(p)) X^j X^k| \\ &\leq C^2 \sup_{x \in K} |R_{\varepsilon jk}(x) - R_{jk} * \rho_\varepsilon(x)|. \end{aligned} \quad (9)$$

Using (5) we conclude from this estimate that, given any $\delta > 0$ we may choose ε_0 such that for all $\varepsilon < \varepsilon_0$, all $p \in K$ and all vectors X with $g_p(X, X) \leq \kappa$ and $\|X\| \leq C$ we have $R_{\varepsilon jk}(p) X^j X^k > -\delta$, which is (3). \square

4 Existence of maximal curves

The next key step in proving the main result is to secure the existence of geodesics maximising the distance to a spacelike hypersurface. To prove this statement we will employ a net \check{g}_ε ($\varepsilon > 0$) of smooth Lorentzian metrics whose lightcones approximate those of g from the inside as in Prop. 3.1. We first need some auxiliary results.

Lemma 4.1. *Let (M, g) be a $C^{1,1}$ -spacetime that is globally hyperbolic. Let h be a Riemannian metric on M and let $K \subseteq M$. Then there exists some $C > 0$ such that the h -length of any causal curve taking values in K is bounded by C .*

Proof. It follows, e.g., from the proof of [25, Lemma 14.13] that (M, g) is non-totally imprisoning, i.e., there can be no inextendible causal curve that is entirely contained in K . Now suppose that, contrary to the claim, there exists a sequence σ_k of causal curves valued in K whose h -lengths tend to infinity. Parametrizing σ_k by h -arclength we may assume that $\sigma_k : [0, a_k] \rightarrow K$, where $a_k \rightarrow \infty$. Also, without loss of generality we may assume that $\sigma_k(0)$ converges to some $q \in K$. Then by [21, Th. 3.1(1)]¹ one may extract a subsequence σ_{k_j} that converges locally uniformly to an inextendible causal curve σ in K , thereby obtaining a contradiction to non-total imprisonment. \square

Lemma 4.2. *Let (M, g) be a globally hyperbolic $C^{1,1}$ -spacetime and let g_ε ($\varepsilon > 0$) be a net of smooth Lorentzian metrics such that g_ε converges locally uniformly to g as $\varepsilon \rightarrow 0$, and let $K \subseteq M$. Then for each $\delta > 0$ there exists some $\varepsilon_0 > 0$ such that for each $\varepsilon < \varepsilon_0$ and each g -causal curve σ taking values in K , the lengths of σ with respect to g and g_ε , respectively, satisfy:*

$$L_g(\sigma) - \delta < L_{g_\varepsilon}(\sigma) < L_g(\sigma) + \delta. \quad (10)$$

Proof. Since $g_\varepsilon \rightarrow g$ uniformly on K , given any $\eta > 0$ there exists some $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and all $X \in TM|_K$ with $\|X\|_h = 1$ we have

$$\|X\|_g - \eta \leq \|X\|_{g_\varepsilon} \leq \|X\|_g + \eta. \quad (11)$$

Consequently, for any $X \in TM|_K$ we have

$$\|X\|_g - \eta \|X\|_h \leq \|X\|_{g_\varepsilon} \leq \|X\|_g + \eta \|X\|_h. \quad (12)$$

Now if $\sigma : [a, b] \rightarrow K$ is any g -causal curve it follows that, for $\varepsilon < \varepsilon_0$,

$$\begin{aligned} L_g(\sigma) - \eta L_h(\sigma) &= \int_a^b \|\sigma'(t)\|_g dt - \eta \int_a^b \|\sigma'(t)\|_h dt \leq \int_a^b \|\sigma'(t)\|_{g_\varepsilon} dt = L_{g_\varepsilon}(\sigma) \\ &\leq L_g(\sigma) + \eta L_h(\sigma). \end{aligned} \quad (13)$$

Finally, by Lemma 4.1 there exists some $C > 0$ such that $L_h(\sigma) \leq C$ for any σ as above. Hence, picking $\eta < \delta/C$ establishes the claim. \square

Proposition 4.3. *Let (M, g) be a future timelike-geodesically complete $C^{1,1}$ -spacetime. Let S be a compact spacelike acausal hypersurface in M , and let $p \in D^+(S) \setminus S$. Then*

- (i) $d_{\check{g}_\varepsilon}(S, p) \rightarrow d(S, p)$ ($\varepsilon \rightarrow 0$).
- (ii) *There exists a timelike geodesic γ perpendicular to S from S to p with $L(\gamma) = d(S, p)$.*

Here we have dropped the subscript from the time separation function $d_g(S, p)$ and the length $L_g(\gamma)$ of the $C^{1,1}$ -metric g to simplify notations. Also we remark that the proof of (i) below neither uses geodesic completeness of M nor compactness of S and hence the \check{g}_ε -distance converges even on general M for any closed spacelike acausal hypersurface S .

¹Note that the required result remains valid for $C^{1,1}$ -metrics (in fact, even for continuous metrics): this follows exactly as in [7, Th. 1.6]

Proof. (i) Since $p \notin S$ we have $c := d(S, p) > 0$. Let $0 < \delta < c$. Then there exists a g -causal curve $\alpha : [0, b] \rightarrow M$ from S to p with $L_g(\alpha) > d(S, p) - \delta$. In particular, α is not a null curve, hence there exist $t_1 < t_2$ such that $\alpha|_{[t_1, t_2]}$ is nowhere null. In what follows we adapt the argument from [6, Lemma 2.4.14] to the present situation. Without loss of generality we may assume that $t_2 = b$. By Theorem A.3 we may find $0 = s_0 < s_1 < \dots < s_N = b$ and totally normal neighbourhoods U_i ($1 \leq i \leq N$) such that $\alpha([s_i, s_{i+1}]) \subseteq U_i$ for $0 \leq i < N$. By Proposition A.6 we obtain that $\alpha(s_{N-1}) \ll \alpha(b)$, hence by Proposition A.4, the radial geodesic σ_N from $\alpha(s_{N-1})$ to p is longer than $\alpha|_{[s_{N-1}, b]}$, and it is timelike. Next, we connect $\alpha(s_{N-2})$ via a timelike radial geodesic σ_{N-1} to some point on σ_N that lies in U_{N-1} . Concatenating σ_{N-1} with σ_N gives a timelike curve longer than $\alpha|_{[s_{N-2}, b]}$. Iterating this procedure we finally arrive at a timelike piecewise geodesic σ from $\alpha(0) = \sigma(0) \in S$ to p of length $L_g(\sigma) \geq L_g(\alpha) > d(S, p) - \delta$.

Since $L_{\check{g}_\varepsilon}(\sigma) \rightarrow L_g(\sigma)$, we conclude that $L_{\check{g}_\varepsilon}(\sigma) > d(S, p) - \delta$ for ε sufficiently small. Moreover, σ is g -timelike and piecewise C^2 , hence is \check{g}_ε -timelike for small ε . Therefore, $d_{\check{g}_\varepsilon}(S, p) \geq L_{\check{g}_\varepsilon}(\sigma) > d(S, p) - \delta$ for ε small.

Conversely, if σ is any \check{g}_ε -causal curve from S to p then σ is also g -causal, hence lies entirely in the set $K := J^-(p) \cap J^+(S, D(S))$. Since $D(S)$ is globally hyperbolic by Theorem A.22 and Proposition A.23, K is compact by Corollary A.29. Then by Lemma 4.2 (applied to the globally hyperbolic spacetime $(D(S), g)$), for ε sufficiently small we have

$$L_{\check{g}_\varepsilon}(\sigma) < L_g(\sigma) + \delta \leq d(S, p) + \delta. \quad (14)$$

Consequently, $d_{\check{g}_\varepsilon}(S, p) \leq d(S, p) + \delta$ for ε sufficiently small. Together with the above this shows (i).

(ii) Since \check{g}_ε has narrower lightcones than g , for each ε the point p lies in $D_{\check{g}_\varepsilon}^+(S) \setminus S$. Also, we may assume ε to be so small that S is \check{g}_ε -spacelike as well as \check{g}_ε -acausal. Then by smooth causality theory (e.g., [25, Th. 14.44]) there exists a \check{g}_ε -geodesic γ_ε that is \check{g}_ε -perpendicular to S and satisfies $L_{\check{g}_\varepsilon}(\gamma_\varepsilon) = d_{\check{g}_\varepsilon}(S, p)$. Let h be some background Riemannian metric on M and let $\gamma_\varepsilon(0) =: q_\varepsilon \in S$, $\gamma'_\varepsilon(0) =: v_\varepsilon$. Without loss of generality we may suppose $\|v_\varepsilon\|_h = 1$. Since $\{v \in TM \mid \pi(v) \in S, \|v\|_h = 1\}$ is compact, there exists a sequence $\varepsilon_j \searrow 0$ such that $q_{\varepsilon_j} \rightarrow q \in S$ and $v_{\varepsilon_j} \rightarrow v \in T_q M$. Denote by γ_v the g -geodesic with $\gamma(0) = q$, $\gamma'(0) = v$. To see that γ is g -orthogonal to S , let $w \in T_q S$ and pick any sequence $w_j \in T_{q_{\varepsilon_j}} S$ converging to w . Then $g(v, w) = \lim \check{g}_{\varepsilon_j}(v_{\varepsilon_j}, w_j) = 0$. Consequently, γ is g -timelike.

Since g is timelike geodesically complete, γ_v is defined on all of \mathbb{R} , so by standard ODE-results (cf., e.g., [16, Sec. 2]) for any $a > 0$ there exists some j_0 such that for all $j \geq j_0$ the curve γ_{ε_j} is defined on $[0, a]$ and $\gamma_{\varepsilon_j} \rightarrow \gamma$ in $C^1([0, a])$ (in fact, it follows directly from this and the geodesic equation that this convergence even holds in $C^2([0, a])$).

For each j , let $t_j > 0$ be such that $\gamma_{\varepsilon_j}(t_j) = p$. Then by (i) we obtain

$$d(S, p) = \lim d_{\check{g}_{\varepsilon_j}}(S, p) = \lim \int_0^{t_j} \|\gamma'_{\varepsilon_j}(t)\|_{\check{g}_{\varepsilon_j}} dt = \lim t_j \|v_{\varepsilon_j}\|_{\check{g}_{\varepsilon_j}} = \|v\|_g \lim t_j, \quad (15)$$

so $t_j \rightarrow \frac{d(S, p)}{\|v\|_g} =: a$. Finally, for j sufficiently large, all γ_{ε_j} are defined on $[0, 2a]$ and we have $p = \gamma_{\varepsilon_j}(t_j) \rightarrow \gamma(a)$, so $p = \gamma(a)$, as well as

$$d(S, p) = \lim \int_0^{t_j} \|\gamma'_{\varepsilon_j}(t)\|_{\check{g}_{\varepsilon_j}} dt = \int_0^a \|\gamma'(t)\|_g dt = L_0^a(\gamma). \quad (16)$$

□

5 Proof of the main result

To prove Theorem 1.1, we first note that without loss of generality we may assume S to be connected. Moreover, by Theorem A.34 we may also assume S to be achronal, and thereby acausal by Lemma A.30 (replacing, if necessary, M by a suitable Lorentzian covering space \tilde{M} and S by its isometric image \tilde{S} in \tilde{M}). Note that since the light cones of \check{g}_ε approximate those of g from the inside it follows that for ε small S is a spacelike acausal hypersurface with respect to \check{g}_ε as well.

We prove the theorem by contradiction and assume that (M, g) is future timelike geodesically complete. Hence we may apply Proposition 4.3 to obtain (using the notation from the proof of that result) for any $p \in D^+(S) \setminus S$:

- (A) \exists g -geodesic $\gamma \perp_g S$ realising the time separation to p , i.e., $L(\gamma) = d(S, p)$.
- (B) \exists \check{g}_ε -geodesics $\gamma_\varepsilon \perp_{\check{g}_\varepsilon} S$ realising the time separation to p , i.e., $L_{\check{g}_\varepsilon}(\gamma_\varepsilon) = d_{\check{g}_\varepsilon}(S, p)$.
- (C) $\exists \varepsilon_j \searrow 0$ such that $\gamma_{\varepsilon_j} \rightarrow \gamma$ in $C^1([0, a])$ for all $a > 0$ (in fact, even in $C^2([0, a])$).

We proceed in several steps.

Step 1. $D^+(S)$ is relatively compact.

The future convergence of S is given by $\mathbf{k} = 1/(n-1)\text{tr}S_U$, with $S_U(V) = -\nabla_V U$ and U the future pointing g -unit normal on S . Analogously, for each ε_j as in (C) we obtain the future convergence \mathbf{k}_j of S with respect to $\check{g}_{\varepsilon_j}$, and we denote the future-pointing $\check{g}_{\varepsilon_j}$ -unit normal to S and the corresponding shape operator by U_j and S_{U_j} , respectively. By Proposition 3.1 (i), $\mathbf{k}_j \rightarrow \mathbf{k}$ uniformly on S . Let $m := \min_S \text{tr}S_U = (n-1)\min_S \mathbf{k}$, and $m_j := \min_S \text{tr}S_{U_j} = (n-1)\min_S \mathbf{k}_j$. By assumption, $m > 0$, and by the above we obtain $m_j \rightarrow m$ as $j \rightarrow \infty$.

Let

$$b := \frac{n-1}{m} \quad (17)$$

and assume that there exists some $p \in D^+(S) \setminus S$ with $d(S, p) > b$. We will show that this leads to a contradiction.

Since each γ_{ε_j} as in (C) is maximising until $p = \gamma_{\varepsilon_j}(t_j)$, it contains no $\check{g}_{\varepsilon_j}$ -focal point to S before t_j . Setting $\tilde{t}_j := (1 - \frac{1}{j})t_j$ it follows that $\exp_{\check{g}_{\varepsilon_j}}^\perp$ is non-singular on $[0, \tilde{t}_j]\gamma_{\varepsilon_j}'(0) = [0, \tilde{t}_j]v_{\varepsilon_j}$. As this set is compact there exist open neighbourhoods W_j of $[0, \tilde{t}_j]v_{\varepsilon_j}$ in the normal bundle $N_{\check{g}_{\varepsilon_j}}(S)$ and V_j of $\gamma_{\varepsilon_j}([0, \tilde{t}_j])$ in M such that $\exp_{\check{g}_{\varepsilon_j}}^\perp : W_j \rightarrow V_j$ is a diffeomorphism. Due to $D_{\check{g}_{\varepsilon_j}}(S)$ being open, we may also assume that $V_j \subseteq D_{\check{g}_{\varepsilon_j}}(S)$.

On V_j we introduce the Lorentzian distance function $r_j := d_{\check{g}_{\varepsilon_j}}(S, \cdot)$ and set $X_j := -\text{grad}(r_j)$. Denote by $\tilde{\gamma}_j$ the re-parametrisation of γ_{ε_j} by $\check{g}_{\varepsilon_j}$ -arclength:

$$\tilde{\gamma}_j : [0, \tilde{t}_j\|v_{\varepsilon_j}\|_{\check{g}_{\varepsilon_j}}] \rightarrow M \quad \tilde{\gamma}_j(t) := \gamma_{\varepsilon_j}(t/\|v_{\varepsilon_j}\|_{\check{g}_{\varepsilon_j}}). \quad (18)$$

Then since $\tilde{\gamma}_j$ is maximising from S to p in $D_{\check{g}_{\varepsilon_j}}^+(S)$, hence in particular in $V_j \cap J_{\check{g}_{\varepsilon_j}}^+(S)$, it follows that $X_j(\tilde{\gamma}_j(t)) = \tilde{\gamma}_j'(t)$ for all $t \in [0, \tilde{t}_j\|v_{\varepsilon_j}\|_{\check{g}_{\varepsilon_j}}]$. Next we define the shape operator corresponding to the distance function r_j by $S_{r_j}(Y) := \nabla_Y^{\check{g}_{\varepsilon_j}}(\text{grad}(r_j))$ for $Y \in \mathfrak{X}(V_j)$. Then $S_{r_j}|_{S \cap V_j} = S_{U_j}|_{S \cap V_j}$ and the expansion $\tilde{\theta}_j := -\text{tr}S_{r_j}$ satisfies the Raychaudhuri equation (cf., e.g., [24])

$$X_j(\tilde{\theta}_j) + \text{tr}(S_{r_j}^2) + \text{Ric}_{\check{g}_{\varepsilon_j}}(X_j, X_j) = 0 \quad (19)$$

on V_j . Consequently, we obtain for $\theta_j(t) := \tilde{\theta}_j \circ \tilde{\gamma}_j(t)$:

$$\frac{d(\theta_j^{-1})}{dt} \geq \frac{1}{n-1} + \frac{1}{\theta_j^2} \text{Ric}_{\tilde{g}_{\varepsilon_j}}(\tilde{\gamma}'_j, \tilde{\gamma}'_j). \quad (20)$$

Now since by (C) the $\tilde{\gamma}_j$ converge in C^1 to the g -timelike geodesic γ , it follows that there exist $\kappa < 0$ and $C > 0$ such that for all j sufficiently large we have $g(\tilde{\gamma}'_j(t), \tilde{\gamma}'_j(t)) \leq \kappa$ as well as $\|\tilde{\gamma}'_j(t)\|_h \leq C$ for all $t \in [0, \tilde{t}_j \|v_{\varepsilon_j}\|_{\tilde{g}_{\varepsilon_j}}]$.

We are therefore in the position to apply Lemma 3.2 to obtain that, for any $\delta > 0$,

$$\frac{d(\theta_j^{-1})}{dt} > \frac{1}{n-1} - \frac{\delta}{\theta_j^2} \quad (21)$$

for j large enough. Pick any c with $b < c < d(S, p)$ and fix $\delta > 0$ so small that

$$b < \frac{n-1}{\alpha m} < c, \quad (22)$$

where m is as in (17) and $\alpha := 1 - (n-1)m^{-2}\delta$. Analogously, let $\alpha_j := 1 - (n-1)m_j^{-2}\delta$, so that $\alpha_j \rightarrow \alpha$ as $j \rightarrow \infty$. Setting $d_j := \tilde{t}_j \|v_{\varepsilon_j}\|_{\tilde{g}_{\varepsilon_j}}$, θ_j is defined on $[0, d_j]$. Note that, for j large, (22) implies the right hand side of (21) to be strictly positive at $t = 0$. Thus θ_j^{-1} is initially strictly increasing and $\theta_j(0) < 0$, so (21) entails that $\theta_j^{-1}(t) \in [-m_j^{-1}, 0)$ on its entire domain. From this we conclude that θ_j has no zero on $[0, d_j]$, i.e., that θ_j^{-1} exists on all of $[0, d_j]$. It then readily follows, again using (21), that $\theta_j^{-1}(t) \geq f_j(t) := -m_j^{-1} + t \frac{\alpha_j}{n-1}$ on $[0, d_j]$. Hence θ_j^{-1} must go to zero before f_j does, i.e., $\theta_j^{-1}(t) \rightarrow 0$ as $t \nearrow T$ for some positive $T \leq \frac{n-1}{\alpha_j m_j}$.

Here we note that due to $\lim d_j = \lim t_j \|v_{\varepsilon_j}\|_{\tilde{g}_{\varepsilon_j}} = d(S, p)$, for j sufficiently large we have by (22)

$$\frac{n-1}{\alpha_j m_j} < c < d_j. \quad (23)$$

This, however, means that $\theta_j^{-1} \rightarrow 0$ within $[0, d_j]$, contradicting the fact that θ_j is smooth, hence bounded, on this entire interval.

Together with (A) this implies that $D^+(S)$ is contained in the compact set $\beta(S \times [0, b])$ where

$$\beta : S \times [0, b] \rightarrow M, \quad (q, t) \mapsto \exp^g(tU(q)), \quad (24)$$

Hence also the future Cauchy horizon $H^+(S) = \overline{D^+(S)} \setminus I^-(D^+(S))$ is compact.

From here, employing the causality results developed in Appendix A, we may conclude the proof exactly as in [25, Th. 14.55B]. For completeness, we give the full argument.

Step 2. The future Cauchy horizon of S is nonempty.

Assume to the contrary that $H^+(S) = \emptyset$. Then $I^+(S) \subseteq D^+(S)$: for $p \in S$, a future-directed timelike curve γ starting at p lies initially in $D^+(S)$ (using Proposition A.23, or Lemma A.25). Hence if γ leaves $D^+(S)$, it must meet $\partial D^+(S)$ and by Lemma A.14 it also meets $H^+(S)$ (since S is achronal it can't intersect S again). But then $H^+(S)$ wouldn't be empty, contrary to our assumption. Hence $I^+(S) \subseteq D^+(S)$. By Step 1, then, $I^+(S) \subseteq \{p \in M \mid d(S, p) \leq b\}$ and hence $L(\gamma) \leq b$ for any timelike future-directed curve emanating from S , which is a contradiction to timelike geodesic completeness of M .

Step 3. The following extension of (A) holds:

(A') $\forall q \in H^+(S) \exists g$ -geodesic $\gamma \perp_g S$ realising the time separation and $L(\gamma) = d(S, q) \leq b$.

Consider the set $B \subseteq N(S)$ consisting of the zero section and all future pointing causal vectors v with $\|v\| \leq b$. B is compact by the compactness of S .

By definition there is a sequence q_k in $D^+(S)$ that converges to q . For any q_k there is a geodesic as in (A) and hence a vector $v_k \in B$ with $\exp(v_k) = q_k$. By the compactness of B we may assume that $v_k \rightarrow v$ for some $v \in B$ and hence by continuity $q_k \rightarrow \exp(v)$. Moreover, we have by construction that $\|v_k\| = d(S, q_k)$. Since d is lower semicontinuous (Lemma A.16), $\|v\| \geq d(S, q)$.

As γ_v is perpendicular to S , hence timelike, our completeness assumption implies that it is defined on $[0, 1]$. Thus it runs from S to q and has length $\|v\|$, which implies $d(S, q) = \|v\| \leq b$.

Step 4. The map $p \mapsto d(S, p)$ is strictly decreasing along past pointing generators of $H^+(S)$.

By Proposition A.24 (iii), $H^+(S)$ is generated by past-pointing inextendible null geodesics. Suppose that $\alpha : I \rightarrow M$ is such a generator, and let $s, t \in I$, $s < t$. Using (A') we obtain a past pointing timelike geodesic γ from $\alpha(t)$ to $\gamma(0) \in S$ of length $d(S, \alpha(t))$. Then arguing as in the proof of Proposition 4.3 (i) we may construct a timelike curve σ from $\alpha(s)$ to $\gamma(0)$ that is strictly longer than the concatenation of $\alpha|_{[s, t]}$ and γ . Therefore,

$$d(S, \alpha(s)) \geq L(\sigma) > L(\alpha|_{[s, t]} + \gamma) = L(\gamma) = d(S, \alpha(t)). \quad (25)$$

Step 5. (M, g) is *not* future timelike geodesically complete.

By step 1, $H^+(S)$ is compact and by Lemma A.16 $p \mapsto d(S, p)$ is lower semicontinuous, hence attains a finite minimum at some point q in $H^+(S)$. But then taking a past pointing generator of $H^+(S)$ emanating from q according to Proposition A.24 (iii) gives a contradiction to step 4. \square

Appendix A: Results from $C^{1,1}$ -causality theory

In this appendix we collect those results on the causality of $C^{1,1}$ -metrics that are used in the main text, that is, A.3, A.4, A.6, A.14, A.16, A.22–A.25, A.29, A.30, A.34, as well as those supplementary statements that are used to prove these, or to secure the compatibility with [13] as explained in Section 2 (A.12 and A.31). Using the results on basic causality theory of $C^{1,1}$ -metrics established in [7, 22, 16, 17], see Theorem A.1 to Lemma A.8 below, combined with the standard proofs in the smooth case, it is a routine matter to prove the remaining results. So instead of providing full proofs we accurately collect all facts and previous statements entering the respective proofs. In this way we provide a concise chain of arguments on the one hand establishing the results and on the other hand showing at which places regularity issues have to be taken into account. Our presentation is essentially based on the one of [25].

We first recall a few fundamental results from $C^{1,1}$ -causality theory that are used throughout the proofs of this section. From now (unless explicitly stated otherwise) we will exclusively work on a $C^{1,1}$ -spacetime (M, g) . Denoting by $\tilde{Q} : T_p M \rightarrow \mathbb{R}$, $v \mapsto g_p(v, v)$ the quadratic form on the tangent space of a Lorentzian manifold, we have:

Theorem A.1. *Let (M, g) be a $C^{1,1}$ -spacetime, and let $p \in M$. Then p has a basis of normal neighbourhoods U , $\exp_p : \tilde{U} \rightarrow U$ a bi-Lipschitz homeomorphism, such that:*

$$\begin{aligned} I^+(p, U) &= \exp_p(I^+(0) \cap \tilde{U}) \\ J^+(p, U) &= \exp_p(J^+(0) \cap \tilde{U}) \\ \partial I^+(p, U) &= \partial J^+(p, U) = \exp_p(\partial I^+(0) \cap \tilde{U}) \end{aligned}$$

Here, $I^+(0) = \{v \in T_p M \mid \tilde{Q}(v) < 0\}$, and $J^+(0) = \{v \in T_p M \mid \tilde{Q}(v) \leq 0\}$. In particular, $I^+(p, U)$ (respectively $J^+(p, U)$) is open (respectively closed) in U .

For a proof, see [22, Th. 1.23] or [17, Th. 3.9].

Corollary A.2. *Let $U \subseteq M$ be open, $p \in U$. Then the sets $I^+(p, U)$, $J^+(p, U)$ remain unchanged if Lipschitz curves are replaced by piecewise C^1 curves, or in fact by broken geodesics.*

See [22, Th. 1.27] or [17, Cor. 3.10].

The usual convexity properties also hold for $C^{1,1}$ -metrics: If U is a normal neighbourhood of each of its points then it is called totally normal or (geodesically) convex. Any pair of its points can then be connected by a unique geodesic contained in U . The following result ([16, Th. 4.1], [22, Th. 1.16]) guarantees existence of such neighbourhoods:

Theorem A.3. *Let M be a smooth manifold with a $C^{1,1}$ -pseudo-Riemannian metric g . Then each point $p \in M$ possesses a basis of totally normal neighbourhoods.*

Concerning curve-lengths in normal neighbourhoods, [22, Th. 1.23] gives:

Proposition A.4. *Let U be a normal neighborhood of $p \in M$. If $p \ll q$ for a point $q \in U$, then the radial geodesic segment σ is the unique longest timelike curve in U connecting p and q .*

The following result provides more information about causal curves intersecting the boundary of $J^+(p, U)$:

Corollary A.5. *Let U be as in Th. A.1, suppose that $\alpha : [0, 1] \rightarrow U$ is causal and $\alpha(1) \in \partial J^+(p, U)$. Then α lies entirely in $\partial J^+(p, U)$ and there exists a reparametrisation of α as a null-geodesic segment.*

See [22, Th. 1.23] or [17, Cor. 3.11].

The following fundamental push-up principle ([7, Lemma 1.22]) in fact even holds for Lipschitz (or, more generally, causally plain continuous) metrics:

Proposition A.6. *Let g be a $C^{0,1}$ -metric on M and let $p, q, r \in M$ with $p \leq q$ and $q \ll r$ or $p \ll q$ and $q \leq r$. Then $p \ll r$.*

Proposition A.7. *Let $U \subseteq M$ as in Th. A.1 be totally normal.*

- (i) *Let $p, q \in U$. Then $q \in I^+(p, U)$ (resp. $\in J^+(p, U)$) if and only if $\vec{pq} := \exp_p^{-1}(q)$ is future-directed timelike (resp. causal). Also, $(p, q) \mapsto \vec{pq}$ is continuous.*
- (ii) *$J^+(p, U)$ is the closure of $I^+(p, U)$ relative to U .*

(iii) The relation \leq is closed in $U \times U$.

(iv) If K is a compact subset of U and $\alpha : [0, b) \rightarrow K$ is causal, then α can be continuously extended to $[0, b]$.

For a proof, see [17, Prop. 3.15].

Lemma A.8. *The relation \ll is open. Moreover, for $A \subseteq U \subseteq M$, where U is open, we have:*

$$\begin{aligned} I^+(A, U) &= I^+(I^+(A, U)) = I^+(J^+(A, U)) = J^+(I^+(A, U)) \\ &\subseteq J^+(J^+(A, U)) = J^+(A, U) \end{aligned} \tag{26}$$

See [17, Cor. 3.12, Cor. 3.13].

Lemma A.9. *Let $S \subseteq M$ be achronal. Then:*

- (i) $S \subseteq D^\pm(S) \subseteq S \cup I^\pm(S)$
- (ii) $D^+(S) \cap I^-(S) = \emptyset$
- (iii) $D^+(S) \cap D^-(S) = S$
- (iv) $D(S) \cap I^\pm(S) = D^\pm(S) \setminus S$.

As in the smooth case, these properties are immediate from the definitions.

Lemma A.10. *Let S be a closed set and let γ be a past inextendible causal curve starting at p that does not meet S . Then:*

- (i) *For any $q \in I^+(p, M \setminus S)$ there exists a past inextendible timelike piecewise geodesic $\tilde{\gamma}$ starting at q that does not meet S ;*
- (ii) *If γ is not a null geodesic, there exists a past inextendible timelike piecewise geodesic $\tilde{\gamma}$ starting at p that does not meet S .*

The proof of the first statement carries over from the smooth case, see [25, Lemma 14.30], using Proposition A.6. For the second statement (to avoid the variational calculus-based proof of [25, Lemma 14.30]) we need the following argument:

Lemma A.11. *Let S be a closed set and let $\alpha : [0, \infty) \rightarrow M \setminus S$ be a past directed causal curve which is not a null geodesic. Then there exists a $\delta > 0$ such that $\alpha(a) \ll \alpha(0)$ (with \ll the relation on $M \setminus S$).*

Proof. Suppose to the contrary that there is no point on the curve α which can be timelike related to $\alpha(0)$ within $M \setminus S$. Using Theorem A.3 we can cover α by totally normal neighbourhoods U_i with $U_i \subseteq M \setminus S$ since $M \setminus S$ is open. Let $t_0 = 0 < t_1 < t_2 \dots$ such that $\alpha|_{[t_i, t_{i+1}]} \subseteq U_{i+1}$. By our assumption, it follows that $\alpha|_{[t_0, t_1]}$ lies in $\partial J^-(\alpha(0), U_1)$. Hence, by Corollary A.5, $\alpha|_{[t_0, t_1]}$ is a null geodesic. Iterating this procedure we obtain that α is a null geodesic, a contradiction. \square

Using this, the proof of Lemma A.10 can be concluded as in [25, Lemma 14.30].

Lemma A.12. *Let S be a closed achronal hypersurface. Then the Cauchy development defined with Lipschitz curves, $D^+(S)$, coincides with the one defined with piecewise C^1 -curves, $D_{C^1}^+(S)$.*

Proof. Obviously, $D^+(S) \subseteq D_{C^1}^+(S)$. Now suppose there existed some $p \in D_{C^1}^+(S) \setminus D^+(S)$. Then there would exist a past inextendible Lipschitz causal curve γ from p such that $\gamma \cap S = \emptyset$. By Theorem A.3, we may cover γ by totally normal neighbourhoods U_1, \dots, U_N, \dots such that $\gamma([s_i, s_{i+1}]) \subseteq U_{i+1}$, $\forall i$. Then we distinguish two cases: If $\gamma([s_i, s_{i+1}]) \subseteq \partial J^+(\gamma(s_i), U_i)$ for all i , then by Corollary A.5 γ is a piecewise null geodesic and therefore piecewise C^1 , a contradiction. The second possibility is that $\exists i, \exists t \in (s_i, s_{i+1})$ such that $\gamma(s_i) \ll \gamma(t)$. But then Lemma A.10 (ii) gives a contradiction. \square

Lemma A.13. *Let S be a closed achronal set. Then $\overline{D^+(S)}$ is the set of all points p such that every past inextendible timelike curve through p meets S .*

This can be shown as in [25, Lemma 14.51], using Theorem A.1, Theorem A.3, Lemma A.9 (i), and Lemma A.10 (i).

Lemma A.14. *Let S be a closed achronal set. Then $\partial D^\pm(S) = S \cup H^\pm(S)$.*

For a proof, follow that of [25, Lemma 14.52], using Lemma A.9 (i), Theorem A.1, Proposition A.6 and Lemma A.13.

Lemma A.15.

- (i) $d(p, q) > 0$ if and only if $p \ll q$
- (ii) If $p \leq q \leq r$, then $d(p, q) + d(q, r) \leq d(p, r)$.

Using Proposition A.6, this follows as in [25, Lemma 14.16].

Lemma A.16. *d is lower semi-continuous.*

This can be proved following [25, Lemma 14.17], using Lemma A.15 and Theorem A.1.

Lemma A.17. *Let $S \subseteq M$ be achronal. Then $\bar{S} \setminus S \subseteq \text{edge}(S)$.*

The proof can be carried out as indicated in the proof of [25, Cor. 14.26], using Theorem A.1, and the fact that the closure of any achronal set S is achronal, which follows from Lemma A.8.

Proposition A.18. *An achronal set S is a topological hypersurface if and only if S contains no edge points.*

For a proof, see [25, Prop. 14.25], employing Theorem A.1.

Corollary A.19. *An achronal set S is a closed topological hypersurface if and only if $\text{edge}(S)$ is empty.*

This can be seen as in [25, Cor. 14.26], using Lemma A.17 and Proposition A.18.

Lemma A.20. *Let $S \subseteq M$ be a Cauchy hypersurface. Then:*

- (i) S is a closed achronal topological hypersurface.

(ii) Every inextendible causal curve intersects S .

To show this one may follow [25, Lemma 14.29], using Lemma A.8 as well as Corollary A.19, and Lemma A.10 (i) (replacing [25, Cor. 14.27]).

Lemma A.21. *Let S be an achronal set and let $p \in D(S)^\circ$. Then every inextendible causal curve through p meets both $I^-(S)$ and $I^+(S)$.*

The proof carries over from [25, Lemma 14.37], and uses Theorem A.1, Lemma A.9, as well as (the proof of) Lemma A.10 (i).

Theorem A.22. *Let S be achronal. Then $D(S)^\circ$ is globally hyperbolic.*

The proof can be done following [25, Th. 14.38]. The constructions used there (limit sequences of causal curves and their properties, existence of convex refinements of open coverings) all carry over to the $C^{1,1}$ -setting, using Theorems A.1, A.3, Propositions A.6, A.7, and Lemmas A.9, A.21.

Proposition A.23. *Let S be a closed acausal topological hypersurface. Then $D(S)$ is open.*

This proposition can be proved following [25, Lemma 14.43], using Theorem A.1, Lemma A.9, Proposition A.6, Proposition A.7 and Proposition A.18.

Proposition A.24. *Let S be a closed acausal topological hypersurface. Then*

- (i) $H^+(S) = I^+(S) \cap \partial D^+(S) = \overline{D^+(S)} \setminus D^+(S)$;
- (ii) $H^+(S) \cap S = \emptyset$
- (iii) $H^+(S)$ is generated by past inextendible null geodesics that are entirely contained in $H^+(S)$.

The proof can be done combining [25, Prop. 14.53] and Theorem A.1, Lemmas A.9, A.10 (ii), A.13, A.14, and Proposition A.23.

Lemma A.25. *Let S be a spacelike hypersurface and let $p \in S$. Then there exists a neighborhood V of p such that $V \cap S$ is a Cauchy hypersurface in V .*

Proof. Let \hat{g}_ε be smooth metrics approximating g from the outside as in Prop. 3.1. Then given any compact neighborhood W of p in M there exists some $\varepsilon > 0$ such that $W \cap S$ is spacelike for \hat{g}_ε . From the smooth theory (e.g., [1, Lemma A.5.6]) we obtain that there exists a neighborhood $V \subseteq W$ such that $V \cap S$ is a Cauchy hypersurface in V for \hat{g}_ε , and consequently also for g . \square

Lemma A.26. *Let S be an achronal set in M and let $p \in D(S)^\circ \setminus I^-(S)$. Then $J^-(p) \cap D^+(S)$ is compact.*

The proof follows [25, Th. 14.40], using Theorem A.1, Proposition A.6, Proposition A.7, Lemma A.9, and Lemma A.21.

Lemma A.27. *Let K be a compact subset of M and let $A \subseteq M$ be such that, $\forall p \in M$, $A \cap J^+(p)$, respectively $A \cap J^-(p)$, is relatively compact in M . Then $A \cap J^+(K)$, respectively $A \cap J^-(K)$, is relatively compact in M .*

The proof can be carried out as in [1, Lemma A.5.3], based on Theorem A.1.

Proposition A.28. *Let $U \subseteq M$ be open and globally hyperbolic. Then the causality relation \leq of M is closed on U .*

This can be proved as in [25, Lemma 14.22] (based on [25, Lemma 14.14]), using Theorem A.3 and Propositions A.4, A.7.

Corollary A.29. *Let S be a Cauchy hypersurface in a globally hyperbolic manifold M and let K be compact in M . Then $S \cap J^\pm(K)$ and $J^\mp(S) \cap J^\pm(K)$ are compact.*

This follows as in [1, Lemma A.5.4], using Proposition A.7, Lemmas A.26, A.27 and Proposition A.28.

We give a proof of the following result, again to avoid the variational calculus-based argument in [25, Lemma 14.42].

Lemma A.30. *Any achronal spacelike hypersurface S is acausal.*

Proof. Let $\alpha : [0, 1] \rightarrow M$ be a future directed causal curve with endpoints $\alpha(0)$ and $\alpha(1)$ in S . If α is not a null-geodesic, by Proposition A.6, we can connect $\alpha(0)$ with $\alpha(1)$ also by a timelike curve, which is a contradiction to the achronality of S . Now let α be a null geodesic. By Lemma A.25, there exists a neighborhood U around $\alpha(0)$ in which $S \cap U$ is a Cauchy hypersurface. Since α is C^2 and causal, it must be transversal to S , so it contains points in $J^+(S, U) \setminus S$. Then we can connect any such point with some point in $S \cap U$ by a timelike curve within U . Concatenating this curve with the remainder of α , we obtain a curve that is not entirely null and meets S twice. As above, this gives a contradiction to achronality. \square

Proposition A.31. *Let S be a spacelike hypersurface in M . Then S is a Cauchy hypersurface if and only if every inextendible causal curve intersects S precisely once.*

Proof. Let S be a Cauchy hypersurface and let α be an inextendible causal curve. By Lemmas A.20(i), A.30, α intersects S at most once. Also, by Lemma A.20 (ii), it has to intersect S at least once, hence the result. \square

The remaining statements in this appendix serve to justify that in the proof of the main result in Section 5 we may without loss of generality assume S to be achronal. This is done using a covering argument, as in [13, 25]. A key ingredient in adapting this construction to the $C^{1,1}$ -setting is the following consequence of [22, Th. 1.39]:

Theorem A.32. *Let M be a smooth manifold with a $C^{1,1}$ -Lorentzian metric and let S be a semi-Riemannian submanifold of M . Then the normal bundle $N(S)$ is Lipschitz. Moreover, there exist neighbourhoods U of the zero section in $N(S)$ and V of S in M such that*

$$\exp^\perp : U \rightarrow V$$

is a bi-Lipschitz homeomorphism.

Lemma A.33. *Let S be a connected closed spacelike hypersurface in M .*

- (i) *If the homomorphism of fundamental groups $i_\# : \pi_1(S) \rightarrow \pi_1(M)$ induced by the inclusion map $i : S \hookrightarrow M$ is onto, then S separates M (i.e., $M \setminus S$ is not connected).*

(ii) If S separates M , then S is achronal.

The proof carries over from [25, Lemma 14.45] using Theorem A.32, Theorem A.1 and a result from intersection theory, namely, that a closed curve which intersects a closed hypersurface S precisely once and there transversally, is not freely homotopic to a closed curve which does not intersect S , cf. [10, p. 78]. The only change to [25, Lemma 14.45] is that for the curve σ we take a geodesic, which automatically is a C^1 -curve (in fact, even C^2), so that the intersection theory argument applies.

Theorem A.34. *Let S be a closed, connected, spacelike hypersurface in M . Then there exists a Lorentzian covering $\rho : \tilde{M} \rightarrow M$ and an achronal closed spacelike hypersurface \tilde{S} in \tilde{M} which is isometric under ρ to S .*

The proof carries over from [25, Prop. 14.48] using Lemma A.33.

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